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Optimization of Biimpulsive Trajectories in the Earth–Moon Restricted Three-Body System

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The problem is addressed of transferring a spacecraft from a low Earth to a low lunar orbit in a planar circular restricted three-body framework. A closed-form approximate expression for the total velocity variation is developed under the assumption of minimum ΔV biimpulsive maneuvers. This approximation quantifies the link between the transfer orbit energy and the minimum ΔV needed to complete the maneuver, but it gives no information on the corresponding mission time. This last problem is addressed in a systematic framework using an optimization process, and the total ΔV is minimized with the constraint that a maximum transfer time is not exceeded. Using a set of mission data taken from the literature, it is found that almost equivalent ΔV and transfer times (compared to a weak stability boundary approach) are obtained without the use of solar perturbations. More important, a consistent methodology is proposed to exploit fully the fundamental tradeoff between the time of flight and the required ΔV.

Nomenclature

G	=	universal gravitational constant
h	=	height
J	=	Jacobi constant
$L_1, \dots, L_5$	=	Lagrange libration points
M	=	main body mass
m	=	spacecraft mass
O	=	center of mass of the two main bodies
P	=	spacecraft position
R	=	circular orbit radius
$R_{\oplus☾}$	=	Earth–Moon distance
$\boldsymbol{r}$	=	position vector
$\mathcal{S}$	=	generic ballistic trajectory
$\mathcal{T}(O; x, y)$	=	rotating reference frame
t	=	time
u, w	=	spacecraft velocity vector components in the $\mathcal{T}$ frame
$\boldsymbol{v}$	=	velocity vector
x, y	=	spacecraft position vector components in the $\mathcal{T}$ frame
$\boldsymbol{x}$	=	state vector
γ	=	flight-path angle
ΔV	=	velocity variation
δ	=	angle between $\boldsymbol{v}_0$ and the x axis
μ	=	Moon's dimensionless mass
ρ	=	spacecraft distance
ω	=	angular velocity

Subscripts

f	=	final
i	=	initial
max	=	maximum
min	=	minimum
t	=	transfer orbit
tot	=	total
0	=	condition at initial time
1	=	condition after the first impulse
2	=	condition before the second impulse
☾	=	moon
$\oplus$	=	Earth

Superscripts

BE	=	bielliptic
BP	=	biparabolic
H	=	Hohmann
WSB	=	weak stability boundaries
$*$	=	minimum ΔV

Introduction

SPACECRAFT trajectories are often characterized by the type of propulsion the spacecraft uses. Basically, one may distinguish between low-thrust and ballistic trajectories. In this paper we consider the latter trajectories, where the name follows from the fact that during the major part of the mission time the spacecraft is only influenced by the gravitational attraction of the various celestial bodies. The spacecraft is propelled by high-thrust engines that impulsively change its velocity at certain time instants.

In recent years the research for new methods of space mission trajectories has received new impetus especially after Belbruno[1] first introduced the concept of weak stability boundaries (WSB). These are regions where the perturbative effects of Earth, moon, and sun on a point mass spacecraft tend to balance. Basically, the transfer between Earth and moon may be divided in two parts. First, the spacecraft is transferred from a parking orbit around the Earth to the WSB of the Earth (under the influence of the sun and moon) via a lunar flyby. Then, with a small amount of energy, the spacecraft reaches the WSB of the moon (under the influence of the sun and Earth) via a ballistic lunar capture trajectory. Belbruno and Miller[2] have shown that substantial improvements in terms of ΔV performance are obtained with respect to Hohmann, biparabolic (BP), and bielliptic (BE) transfer strategies. This method was successfully applied[3] to rescue the Japanese Hiten mission in 1990. Its most significant drawback is that long times of flight are needed,[4] on the order of 3–5 months, for a low Earth orbit (LEO) to low lunar orbit (LLO) transfer. Another approach consists in approximating the sun–Earth–moon-spacecraft four-body system as two coupled three-body systems, using the invariant manifold structures associated with the Lagrange points (see Ref. 5). From a different point

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of view, the Earth-to-moon transfer problem has also been studied using a planar circular restricted three-body model (PCR3BP). Bollt and Meiss[6] considered the transfer from a nearly circular parking orbit of radius 59,669 km to a quasi-periodically processing ellipse around the moon, with a perilune of 13,970 km. Their method exploits the fact that long trajectories in a compact phase space are recurrent. Starting with a long trajectory that eventually reaches the target, they use small, suitably chosen perturbations to find a nearby short path, cutting recurrent loops from the trajectory and reducing the time of flight. They found a trajectory that achieves ballistic capture with a ΔV of 749.6 m/s. This amounts to a 38% ΔV saving when compared to a Hohmann transfer, but requires a transfer time of 747 days.

Later, Schroer and Ott[7] returned to the problem addressed by Bollt and Meiss,[6] using another approach, called the pass targeting method. They found a considerably shorter transfer, requiring 377.5 days with roughly the same total ΔV, 748.9 m/s. In a recent paper, Ross et al.,[8] using the method of Schroer and Ott[7] together with methods for achieving ballistic capture,[9] found a transfer with a time of flight of 65 days, requiring a ΔV of 860.1 m/s. In other words, this last trajectory needs a much lower time of flight when compared to those by Bollt and Meiss and Schroer and Ott, with not much more fuel consumption.

The preceding results may be explained as follows. Once a parking orbit around the Earth has been chosen, both Bollt and Meiss[6] and Schroer and Ott[7] guess that, in contrast to a typical Hohmann transfer, a chaotic orbit should be selected to eliminate the need for a large deceleration at the moon. Because there is a certain required minimum energy $J = J_{\min}$ (where J is the Jacobi constant) for which a transfer to the moon is possible, they choose a value for J that is slightly greater than $J_{\min}$ but smaller than that value for which the trajectories may escape from the Earth–moon system. When the value of J is fixed, which is the same for Bollt and Meiss and Schroer and Ott, the initial ΔV is established, equal to 744.4 m/s. This value represents more than 99% of the total ΔV to complete the mission because the total thrust required for controlling the trajectory and stabilizing it around the moon is very small. (In both cases[6,7] the sum of the two ΔV is around 5 m/s.) Accordingly, the value found by Schroer and Ott is close to the minimum required for a transfer between the two orbits,[8] provided a ballistic capture around the moon is sought. In this context, the result found by Ross et al.[8] is similar in the spirit, the only difference being that recurrent loops are cut from the trajectory with higher ΔV to reduce the total time of flight.

From an engineering viewpoint, the importance of such results should not be overestimated. In fact, the choice of a parking orbit of radius 59,669 km is due to that it is unlikely to reach the moon from a tight Earth parking orbit with an energy state near $J_{\min}$. However, when the ΔV found by Schroer and Ott[7] is summed to the additional ΔV that is necessary to put the spacecraft in the parking orbit of 59,669 km, it easily verified that the total ΔV is not minimum for a complete Earth-to-moon transfer.

For these reasons, in this paper we revisit the Earth-to-moon transfer problem within a PCR3BP formulation using transfer orbits that have energy significantly above $J_{\min}$. By the taking advantage of the fact that trajectories with higher J are less chaotic, a reasonable time of flight may be obtained. In contrast to the earlier described approaches, the problem is now addressed in a systematic framework using an optimization procedure, where the total ΔV is minimized with the constraint that a maximum transfer time is not exceeded. Accordingly, the fundamental trade-off between the time of flight and the required ΔV is taken into account.

The paper is organized as follows. First, an analytical approximation to the total velocity variation is developed under the assumption of minimum ΔV biimpulsive maneuvers. This approximation is a function of the Jacobi constant for the transfer orbit and of the radii of the orbits around the Earth and the moon, but it is independent of the spacecraft initial and final positions along the circular orbits. Then, the minimum ΔV trajectory corresponding to a given value of the Jacobi constant is obtained numerically through an hybrid strategy that combines genetic algorithms with a deterministic simplex method. Finally, a detailed case study is presented. In particular, when a LEO to LLO transfer with the same mission data as Belbruno and Miller[2] is considered, it is found that ballistic trajectories in the PCR3BP model exist requiring ΔV and mission times almost equivalent to those obtained by Yamakawa et al.[3,10] through a WSB approach.

Mathematical Preliminaries

In this paper, a PCR3BP model is considered. As usual, the masses of Earth $M_\oplus$ and the moon $M_☾$ affect the motion of the spacecraft mass m, without being affected by m themselves. The two main bodies $M_\oplus$ and $M_☾$ are assumed to be in circular Keplerian orbits about their mutual center of mass O. The two orbits are coplanar and have the same constant angular velocity ω. The motion of m is confined to stay in the orbital plane of $M_\oplus$ and $M_☾$. A standard canonical system of units associated with this model is used. More precisely, the mass unit (MU) $\triangleq M_\oplus + M_☾ \cong 6.0471 \times 10^{24}$ kg, the distance unit (DU) $\triangleq R_{\oplus☾} \cong 3.844 \times 10^5$ km, and the time unit (TU) $\triangleq \sqrt{[R^3_{\oplus☾}/G(M_\oplus + M_☾)]} \cong 104$ h, where $R_{\oplus☾}$ is the Earth–moon distance and G is the universal gravitational constant. With these units, both the total mass and the angular velocity of the system are normalized to one.

Let $\mathcal{T}(O; x, y)$ be an orthogonal reference frame rotating with the two primaries, with the origin in the center of mass of the system and x axis pointing from Earth to moon. It is assumed (Fig. 1) that the coordinates of $M_\oplus$ are $(-\mu, 0)$ and the coordinates of M_c are $(1 - \mu, 0)$, where

$$\mu \triangleq M_☾/(M_\oplus + M_☾) \cong 0.0123 \tag{1}$$

Let $\boldsymbol{r} \equiv [x, y]^T$ and $\boldsymbol{v} \equiv [u, , w]^T$ be the position and velocity vectors of the spacecraft. The equations of motion are[11]

$$\dot{x} = u \tag{2}$$

$$\dot{y} = w \tag{3}$$

$$\dot{u} = x + 2w - (1 - \mu)\left[(x + \mu)\big/\rho^3_\oplus\right] - \mu\left[(x + \mu - 1)\big/\rho^3_☾\right] \tag{4}$$

$$\dot{w} = -2u + \left\{1 - (1 - \mu)\big/\rho^3_\oplus - \left(\mu\big/\rho^3_☾\right)\right\}y \tag{5}$$

where $\rho_\oplus$ and $\rho_☾$ are the distances of the spacecraft from the Earth and the moon. Referring to Fig. 1, one has

$$\rho_\oplus \triangleq \sqrt{(x + \mu)^2 + y^2} \tag{6}$$

$$\rho_☾ \triangleq \sqrt{(x + \mu - 1)^2 + y^2} \tag{7}$$

To get a more compact form, we define a state vector $\boldsymbol{x} \triangleq [x, y, u, w]^T$. Then Eqs. (2–5) may be represented as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{8}$$

It is well known that five equilibrium points $L_1, \ldots, L_5$ (the Lagrange libration points) exist, where the gravitational and centrifugal

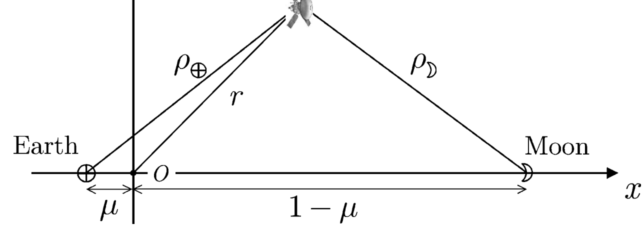
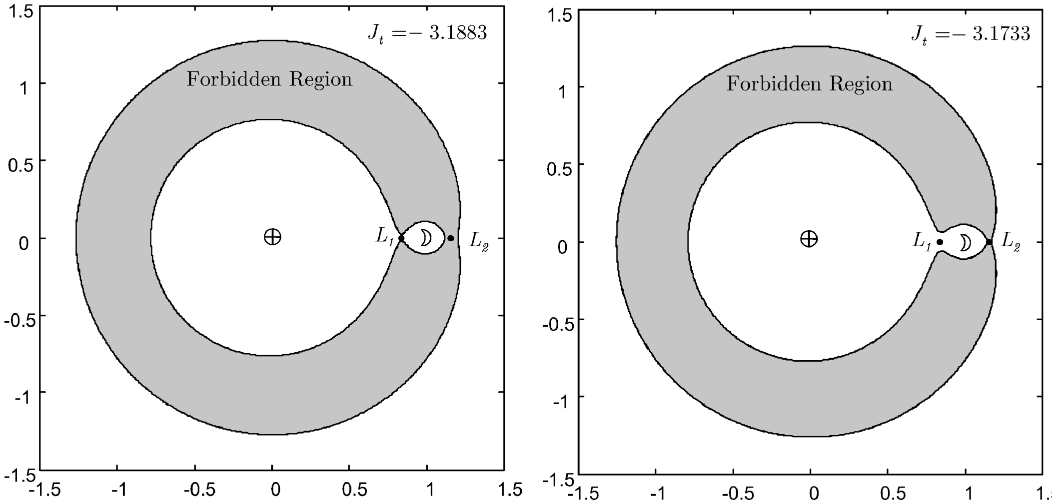


Fig. 1 Rotating reference frame $\mathcal{T}(O; x, y)$.



a) b)

Fig. 2 Hill's region for a) $J_t = J_{L_1} \overset{\Delta}{=} -3.1883$ and b) $J_t = J_{L_2} \overset{\Delta}{=} -3.1733$.

Table 1 Coordinates of Lagrange points in the $\mathcal{T}(O; x, y)$ frame

Coordinate		L_1	L_2	L_3	L_4	L_5
x	(DU)	0.8369	1.1557	−1.0051	0.4879	0.4879
y	(DU)	0	0	0	0.8660	−0.8660

forces acting on the spacecraft are balanced. Because the labeling of the Lagrange points is not consistent across different textbooks, the conventions adopted in this paper have been summarized in Table 1. In particular, point L_1 (the cislunar point) is located on the x axis between the Earth and the moon.

Recall that the Jacobi integral

$$J = u^2 + w^2 - (x^2 + y^2) - 2[(1-\mu)/\rho_\oplus + \mu/\rho_☾] \tag{9}$$

is a constant of motion, whose value is determined from the initial conditions, and is used as an energy measure. Indeed, the larger is $|J|$, the smaller is the spacecraft energy [as is seen by the $\mathcal{T}(O; x, y)$ frame]. Setting $u^2 + w^2 = 0$ in Eq. (9), for a given energy constant J, provides an algebraic expression of the boundary of Hill's region where the spacecraft is energetically permitted to move around. In this paper, we study trajectories characterized by a Jacobi constant above $J_{L_1} = -3.1883$. In fact the condition $J_t > J_{L_1}$ (where J_t is the value of the Jacobi constant for the transfer orbit) opens a corridor for feasible motion between Earth and moon (Fig. 2).

Mission Analysis

Consider the transfer of a spacecraft from an initial (subscript i) circular orbit around the Earth to a final (subscript f) circular orbit around the moon by means of a biimpulsive transfer (subscript t) trajectory. Our aim is to develop an analytical approximation to the total ΔV as a function of the Jacobi constant for the transfer orbit. To this end, let J_i be the Jacobi constant associated with the spacecraft initial state. Because tangential ΔV are optimal (in the sense of minimum total ΔV required) for escape maneuvers,[12] assume that a tangential burn is applied at the beginning of the maneuver and let P_1 be the corresponding spacecraft position. The velocity variation ΔV_1 needed to place the spacecraft into the transfer trajectory is given by

$$\Delta V_1 = \|\boldsymbol{v}_1 - \boldsymbol{v}_i\| = v_1 - v_i \tag{10}$$

where $v_i = \|\boldsymbol{v}_i\| = \sqrt{(u_i^2 + w_i^2)}$ and $v_1 = \|\boldsymbol{v}_1\| = \sqrt{(u_1^2 + w_1^2)}$ are the spacecraft velocities before and after the first impulse has been applied. Note that the spacecraft position does not vary during the impulsive maneuver; Eq. (9) yields

$$J_t - J_i = v_1^2 - v_i^2 \tag{11}$$

When Eq. (10) is substituted into Eq. (11), the following result is obtained:

$$\Delta V_1 = -v_i + \sqrt{v_i^2 + J_t - J_i} \tag{12}$$

In the optimal case, the transfer trajectory is tangent to the final circular orbit at a suitable point P_2. When the spacecraft reaches that position, another tangential burn is applied to circularize the orbit and complete the maneuver. The second velocity variation ΔV_2 is

$$\Delta V_2 = \|\boldsymbol{v}_2 - \boldsymbol{v}_f\| = v_2 - v_f \tag{13}$$

where $v_2 = \|\boldsymbol{v}_2\| = \sqrt{(u_2^2 + w_2^2)}$ and $v_f = \|\boldsymbol{v}_f\| = \sqrt{(u_f^2 + w_f^2)}$ are the spacecraft velocities before and after the second impulse has been applied. In analogy with result (12), the following expression for ΔV_2 is found:

$$\Delta V_2 = -v_f + \sqrt{v_f^2 + J_t - J_f} \tag{14}$$

Therefore, the total velocity variation $\Delta V_{\text{tot}} = \Delta V_1 + \Delta V_2$ is

$$\Delta V_{\text{tot}} = -(v_i + v_f) + \sqrt{v_i^2 + J_t - J_i} + \sqrt{v_f^2 + J_t - J_f} \tag{15}$$

Equation (15) is not fully satisfactory because the link between the total velocity variation and the value of the Jacobi constant for the transfer orbit J_t is given as a function of the spacecraft initial and final positions (through the terms v_i, J_i and v_f, J_f). A remarkable simplification is possible under the hypothesis of low-height initial and final orbits. In fact, assuming R_i and $R_f \ll 1$ DU, where R_i and R_f are the radii of the initial and final orbits, the following result is found (see the Appendix):

$$\Delta V_{\text{tot}} \cong -\left(\sqrt{(1-\mu)/R_i} - R_i + \sqrt{\mu/R_f}\right) + \sqrt{R_i^2 + \mu^2 + 2(1-\mu)/R_i + 2\mu/(1-R_i) - 2R_i\mu + J_t}$$
$$+ \sqrt{(1-\mu)^2 + R_f^2 + (2\mu/R_f) + 2(1-\mu)/(1-R_f) - 2(1-\mu)R_f + J_t} \tag{16}$$

Note that the total velocity variation is approximated through an expression that is independent of the spacecraft initial and final positions along the circular orbits. In other words, once the two circular orbits have been chosen, that is, R_i and R_f have been fixed, the ΔV_{tot} is computed as a function of J_t only. Accordingly, Eq. (16) has interesting consequences. First, it shows that ΔV_{tot} increases with J_t. More important, it defines a lower bound to the admissible ΔV_{tot} for a given transfer orbit. In particular, once the values of R_i and R_f are specified, the minimum ΔV_{tot} is obtained for $J_t = J_{L_1}$. Consider, for instance, two circular orbits at a height $h = 200$ km above the Earth's and moon's surfaces, respectively. Then $R_i = 1.7112 \times 10^{-2}$ DU, $R_f = 5.0468 \times 10^{-3}$ DU, and the minimum ΔV_{tot} compatible with a planar biimpulsive transfer is $\Delta V_{\text{tot}}^{\min} \cong 3.608$ DU/TU.

However, a transfer orbit with $\Delta V_{\text{tot}} = \Delta V_{\text{tot}}^{\min}$ is impractical because the transfer time becomes infinite as $J_t \to J_{L_1}$. (In fact for $J_t = J_{L_1}$ the corridor between Earth and the moon reduces to a point.) The same conclusion applies to missions characterized by excessive transfer times. For this reason, it is important to guarantee that the transfer time of a given mission is less than a maximum allowable value. Unfortunately, a closed-form expression involving the mission time as a function of the Jacobi constant J_t cannot be obtained, and a numerical solution is necessary. This problem is addressed in the next section.

Minimum ΔV Trajectories

Assume that R_i and $R_f \ll 1$ DU are fixed and recall that P_1 and P_2 are the spacecraft positions at the beginning and at the end of the ballistic transfer. For a given value of J_t, we look for the biimpulsive trajectory that minimizes ΔV_{tot}. A generic trajectory $\mathcal{S}(J_t)$ between P_1 and P_2 is obtained through a numerical integration of Eq. (8) provided that the spacecraft position (x_0, y_0) and velocity (u_0, w_0) are given at the initial time $t = t_0$. Note that for a given J_t the value of the spacecraft velocity $v_0 = \|\boldsymbol{v}_0\| = \sqrt{(u_0^2 + w_0^2)}$ is known through Eq. (9). Therefore, the velocity components are more suitably expressed as a function of the angle δ between $\boldsymbol{v}_0$ and the x axis, that is,

$$u_0 = v_0 \cos\delta \tag{17}$$

$$w_0 = v_0 \sin\delta \tag{18}$$

Starting from a point belonging to the circular orbit around the Earth, the minimum ΔV trajectory should minimize the ΔV_{tot} as a function of x_0, y_0, and δ, subject to the constraint $x_0^2 + y_0^2 = R_i^2$. However, in this form the problem is very involved because the solution is extremely sensitive to small variations in the initial conditions. A simpler and more robust approach is obtained starting from that particular point P_0 of the trajectory whose abscissa coincides with the libration point L_1 (Fig. 3). In fact, the spacecraft must pass near

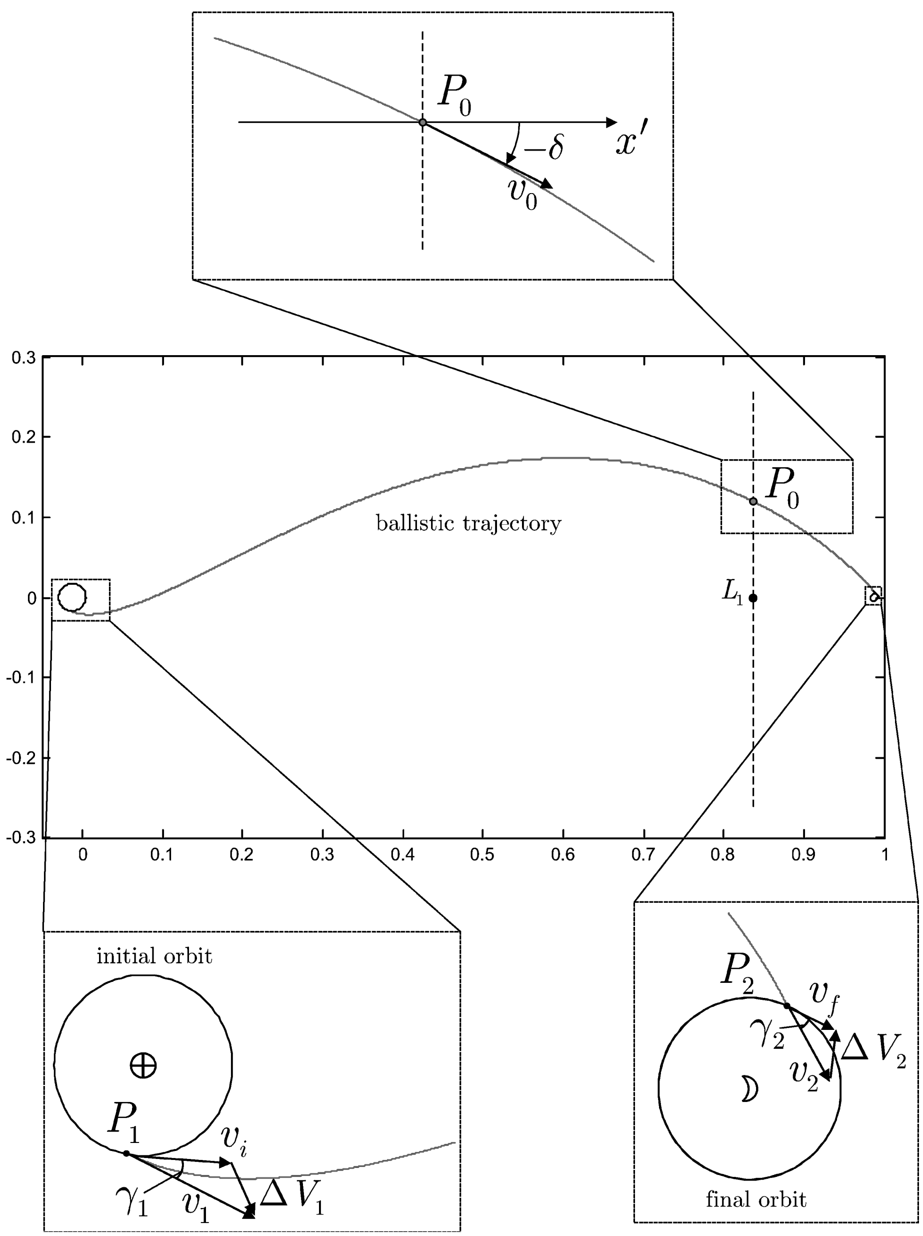


Fig. 3 Generic ballistic trajectory with x' parallel to x axis of the rotating frame.

L_1 to remain in the admissible Hill's region (Fig. 2). With such a choice, the state vector at time t_0 is given by

$$\boldsymbol{x}_0 = [x_{L_1}, y_0, v_0 \cos\delta, v_0 \sin\delta]^T \tag{19}$$

and is a function of only two independent parameters, y_0 and δ. The ballistic trajectory consists of the two arcs $P_1 P_0$ and $P_0 P_2$ corresponding to a terrestrial and a lunar phase trajectory. From P_0, the trajectory arc $P_1 P_0$ is propagated backward, using a numerical integration of Eq. (8), for a time interval $\Delta t = \Delta t_1$ until an intersection with the circular parking orbit around the Earth (point P_1) is found. Clearly, in P_1 both the position $\boldsymbol{r}_1$ and the velocity $\boldsymbol{v}_1$ of the spacecraft (and, hence, the flight-path angle γ_1) are known. Because the local circular speed at radius R_i is given by[12] $v_i = \sqrt{[(1-\mu)/R_i] - R_i}$, the velocity variation ΔV_1 is obtained as

$$\Delta V_1 = \sqrt{v_1^2 + v_i^2 - 2\, v_1\, v_i\, \cos\gamma_1} \tag{20}$$

In a similar way, arc $P_0 P_2$ of the ballistic trajectory is obtained by forward integration of Eq. (8) for a time interval $\Delta t = \Delta t_2$, until an intersection with the circular parking orbit around the moon (point P_2) is found and the corresponding spacecraft velocity $\boldsymbol{v}_2$ as well as the flight-path angle γ_2 are obtained. In this case, the local circular speed at radius R_f is $v_f = \sqrt{(\mu/R_f)}$, and the velocity variation ΔV_2 is given by

$$\Delta V_2 = \sqrt{v_2^2 + v_f^2 - 2\, v_2\, v_f\, \cos\gamma_2} \tag{21}$$

In both cases, a fourth-order Runge–Kutta method with variable step size, absolute and relative tolerances of 10^{-10}, has been used for the numerical integrations.

To summarize, having fixed R_i, R_f, and J_t, the minimum ΔV ballistic trajectory $\mathcal{S}^*(J_t)$ is found by solving the problem

$$\Delta V_{\text{tot}}\left(y_0^*, \delta^*, J_t\right) = \min_{y_0, \delta} \Delta V_{\text{tot}}(y_0, \delta, J_t) \tag{22}$$

When the problem (22) is solved, the corresponding mission time $\Delta t^* \triangleq \Delta t(\mathcal{S}^*(J_t))$ can be calculated. Of course, not all of the minimum ΔV trajectories are of practical use because a reduction in ΔV_{tot} implies an increase in Δt. Accordingly, we will consider only those trajectories whose mission time is less than a maximum, that is, $\Delta t^* \leq \Delta t^{\max}$.

Problem (22) is solved numerically using a two-stage strategy that splits the optimization procedure in two steps: First, a genetic algorithm[13] explores a large search space to localize the global minimum ΔV region. Then, a deterministic Nelder–Mead simplex method (see Ref. 14) is employed to reach the minimum accurately. More complex hybridization methods are available in the literature,[15] but the preceding choice offers a good tradeoff between reliability and complexity. Once a numerical solution to the problem is found, one may wonder whether the algorithm has indeed generated a global or a local minimum solution. Although it may be difficult to guarantee that a global minimum is obtained, in this case there are two checks that can help validate the results. First, the trajectory should be tangent to both the initial and final circular orbits. Second, and more important, the minimum of ΔV_{tot} is known. In fact, it is given by Eq. (15) and is approximated through Eq. (16). What is not known is the trajectory $\mathcal{S}^*(J_t)$ that corresponds to the minimum of $\Delta V_{\text{tot}}(J_t)$. The two-stage optimization strategy gives solutions that satisfy both the checks. As a result, the described methodology allows one to establish a correspondence between the minimum ΔV_{tot} and the corresponding mission times and to investigate the tradeoff between the time of flight and the required total velocity variation.

Case Study

For comparative purposes, we use the mission data taken from Belbruno and Miller.[2] The problem is to find the minimum ΔV ballistic trajectories that transfer a spacecraft from a low Earth circular orbit at 167-km altitude ($R_i = 1.7026 \times 10^{-2}$ DU) to a low circular orbit around the moon of 100-km altitude ($R_f = 4.7866 \times 10^{-3}$ DU). The earlier described optimization strategy has been applied to solve problem (22) for different values of the Jacobi constant in the range $[-2.757, -1.556]$ and a maximum mission time $\Delta t^{\max} = \Delta t^{\text{WSB}} = 32.31$TU (140 days). A comparison between the exact ΔV_{tot}, that is, obtained solving the optimization problem, and the approximate value given by Eq. (16) is shown in Fig. 4 as a function of J_t. Note that Eq. (16) gives a very accurate estimate of ΔV_{tot} and that a nearly linear relationship exists for J_t vs ΔV_{tot}. This is due to the limited variation range of J_t. The rationale for such a small variation range of the Jacobi constant is that values of J_t outside the chosen interval are impractical because they correspond to too long mission times (for $J_t < -2.757$) or too large values of ΔV (for $J_t > -1.556$).

Figure 4 also shows the ΔV for the four cases BP, BE, Hohmann, and WSB (from Ref. 2), and the corresponding mission times are compared in Table 2. The minimum $\Delta V_{\text{tot}} = 3.748$ DU/TU (3838 m/s) is obtained with a WSB technique, with a

Table 2 Performance of WSB, BP, Hohmann (H), and BE transfers for $R_i = 1.7026 \times 10^{-2}$ DU (6 545 km) and $R_f = 4.7866 \times 10^{-3}$ DU (1 840 km), from Ref. 2

	ΔV_{tot}		Δt	
Transfer type	m/s	DU/TU	Days	TU
WSB	3 838	3.748	140	32.31
BP	3 953	3.860	∞	∞
H	3 991	3.897	5	1.15
BE[a]	4 148	4.051	90	20.86

[a]Apogee distance: 1.5×10^6 km.

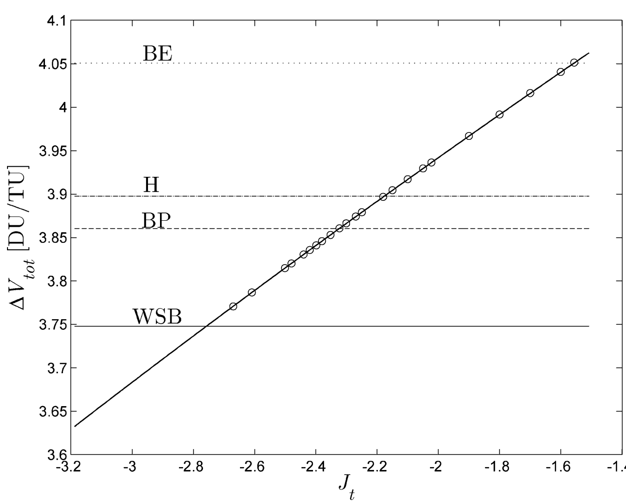


Fig. 4 Comparison between the ○, minimum numerical ΔV_{tot} and ——, approximate analytical values given by Eq. (16).

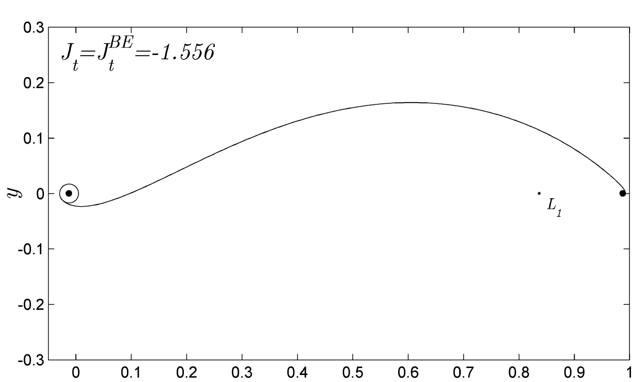


Fig. 5 Ballistic transfer trajectory for $J_t = J_t^{\text{BE}} = -1.556$ in rotating frame.

transfer time of 32.31 TU (140 days). From Fig. 4, it is clear that it is possible to obtain ballistic trajectories that are equivalent, from an energetic viewpoint, to these four cases. In fact, to obtain a ballistic trajectory with $\Delta V_{\text{tot}} = \Delta V_{\text{tot}}^{\text{H}} = 3.897$ DU/TU (corresponding to a Hohmann transfer), it is sufficient to choose $J_t = J_t^H = -2.177$. Likewise, from Fig. 4 it is found that $J_t^{\text{BP}} = -2.323$, $J_t^{\text{BE}} = -1.556$, and $J_t^{\text{WSB}} = -2.757$ for BP, BE, and WSB transfers, respectively. Note that the interval of variation of J_t chosen for the simulations corresponds to $J_t \in [J_t^{\text{WSB}}, J_t^{\text{BE}}]$. The ballistic trajectories for the cases J_t^{BE}, J_t^H, and J_t^{BP} are shown in Figs. 5–7.

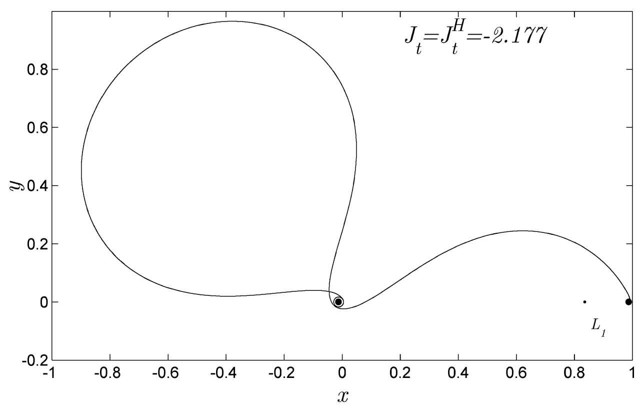


Fig. 6 Ballistic transfer trajectory for $J_t = J_t^H = -2.177$ in rotating frame.

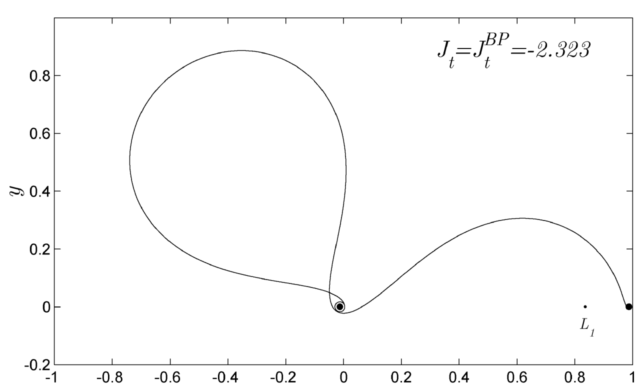


Fig. 7 Ballistic transfer trajectory for $J_t = J_t^{\text{BP}} = -2.323$ in rotating frame.

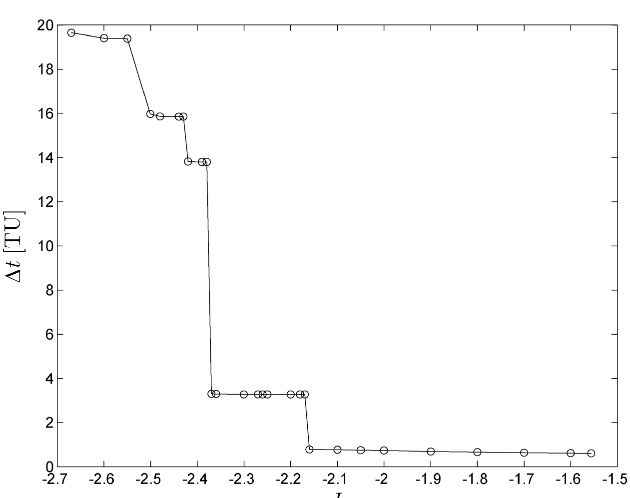


Fig. 8 Mission times obtained through optimization process.

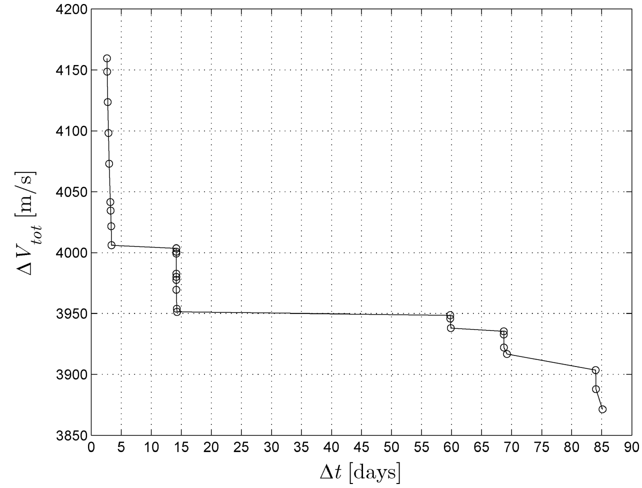


Fig. 9 ΔV_{tot} vs mission time.

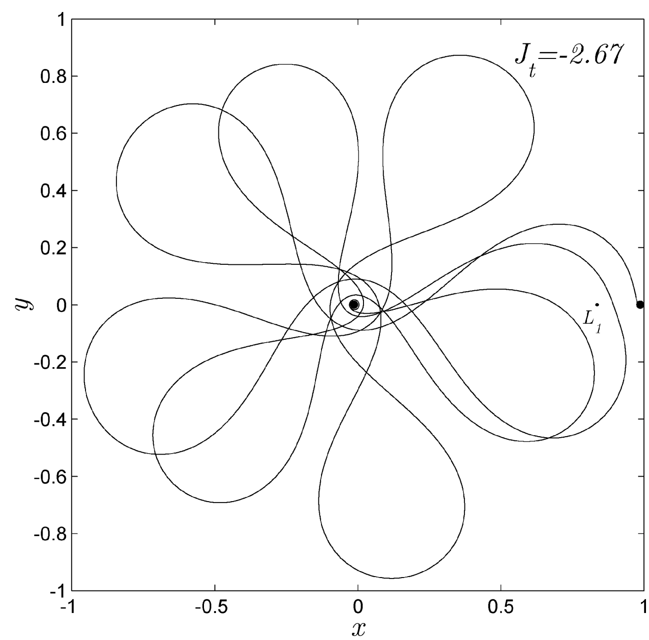


Fig. 10 Ballistic transfer trajectory for $J_t = -2.67$ in rotating frame.

The effectiveness of the numerical method is confirmed by the fact that the trajectories are tangent to both the Earth and the moon circular orbits, as expected. The mission times are shown in Fig. 8 as a function of J_t. Finally, the ΔV_{tot} vs the mission time is shown in Fig. 9. From Fig. 8, it turns out that sudden increases of mission times may occur with small variations of J_t. This is due to the chaotic behavior of the system.[5] Also note that the minimum value of J_t consistent with the constraint $\Delta t \leq \Delta t^{\text{WSB}}$ is $J_t = -2.67$. The corresponding total velocity variation required is $\Delta V_{\text{tot}} = 3.7707$ DU/TU (3861 m/s) and the total mission time is 19.65 TU (85 days). This result is very interesting because it shows that ballistic trajectories in the PCR3BP model exist whose mission time is nearly identical to the value obtained by Yamakawa et al.[3,10] using a WSB approach. Also, for $J_t = -2.67$, a reduction of the mission time of 39% (when compared to the result by Belbruno and Miller[2]) is obtained at the expense of an increase of only 0.0227 DU/TU (23.2 m/s) in the ΔV_{tot}. The corresponding ballistic trajectory is shown in Fig. 10.

Conclusions

The problem of transferring a spacecraft from a low Earth to a low lunar circular orbit using a PCR3BP model has been investigated. An analytical approximation to the total velocity variation has been developed under the assumption of minimum ΔV biimpulsive maneuvers. This approximation is a function of the Jacobi constant for the transfer orbit and of the radii of the orbits around the Earth and the moon, but it is independent of the spacecraft initial and final positions along the circular orbits. The approximation quantifies the link between the transfer orbit energy (through the value of J_t) and the minimum ΔV needed to complete the maneuver, but it gives no information on the corresponding mission time. This information is obtained by numerically solving an optimization problem in which the minimum ΔV trajectory for a given value of J_t is computed. A hybrid methodology has been employed that combines genetic algorithms (localizing the global minimum region) with a deterministic simplex method that refines the solution. Because of the chaotic behavior of the system, sudden increases of mission times may occur with small variations of J_t as J_t is reduced.

When a set of mission data taken from Belbruno and Miller[2] is assumed and the results compared with a WSB approach,[3,10] it is found that almost equivalent ΔV and transfer times are obtained without the use of solar perturbation, as is employed in the WSB method. More important, a consistent methodology is proposed that takes into account the fundamental tradeoff between the time of flight and the required ΔV. These results may be effectively used in a preliminary phase of the mission design. In fact, the problem of accurately predicting the orbit of a spacecraft in the Earth–moon space requires the considerations of other effects such as the lunar orbit eccentricity, the sun's gravitational attraction, and the solar radiation pressure. This is left for future work.

Appendix: Analytical Approximate Expression of ΔV_{tot}

The minimum total velocity variation ΔV_{tot} needed to transfer the spacecraft from an initial circular orbit around the Earth to a final circular orbit around the moon by means of a biimpulsive transfer trajectory is given by Eq. (15), where

$$v_i \cong \sqrt{(1-\mu)/R_i} - R_i \tag{A1}$$

$$v_f \cong \sqrt{\mu/R_f} \tag{A2}$$

When Eq. (9) is used, the following expressions are obtained:

$$v_i^2 - J_i = \left(x_i^2 + y_i^2\right) + 2\left[(1-\mu)/R_i + \mu/\rho_{☾_i}\right] \tag{A3}$$

$$v_f^2 - J_f = \left(x_f^2 + y_f^2\right) + 2\left[(1-\mu)/\rho_{\oplus_f} + \mu/R_f\right] \tag{A4}$$

where $\rho_{☾_i}$ and $\rho_{\oplus_f}$ are the distances of the spacecraft from the moon at the beginning of the transfer and from the Earth at the end of the transfer. With simple geometrical considerations it is found that

$$\left(x_i^2 + y_i^2\right) = R_i^2 + \mu^2 - 2\,R_i\,\mu\,\cos l_\oplus \tag{A5}$$

$$\left(x_f^2 + y_f^2\right) = (1-\mu)^2 + R_f^2 + 2\,(1-\mu)\,R_f\,\cos l_☾ \tag{A6}$$

$$\rho_{☾_i} = \sqrt{R_i^2 + 1 - 2\,R_i\,\cos l_\oplus} \tag{A7}$$

$$\rho_{\oplus_f} = \sqrt{R_f^2 + 1 + 2\,R_f\,\cos l_☾} \tag{A8}$$

where $l_\oplus$ and $l_☾$ are the true longitude of the satellite (relative to the positive x axis of the rotating frame) in the initial and final orbit, respectively. When Eqs. (A1–A4) are substituted into Eq. (15) and Eqs. (A5–A8) are kept in mind, an expression for ΔV_{tot} is found in the form

$$\Delta V_{tot} = \Delta V_{tot}(R_i, R_f, l_\oplus, l_☾, J_t) \tag{A9}$$

Once R_i, R_f, and J_t are given, ΔV_{tot} is function of the initial and final spacecraft position through the angles $l_\oplus$ and $l_☾$. However, when it is assumed that R_i and $R_f \ll 1$ DU, it is possible to neglect the dependence on those angles. In fact, the maximum and minimum values of ΔV_{tot} are obtained for $\cos l_\oplus = 1$ and $\cos l_☾ = -1$ and $\cos l_\oplus = R_i/2$ and $\cos l_☾ = -R_f/2$, respectively. The corresponding expressions for ΔV_{tot} are

$$\Delta V_{tot}^{max}(R_i, R_f, J_t) = -\left(\sqrt{(1-\mu)/R_i} - R_i + \sqrt{\mu/R_f}\right) + \sqrt{R_i^2 + \mu^2 + 2\,(1-\mu)/R_i + 2\,\mu/(1-R_i) - 2\,R_i\,\mu + J_t}$$
$$+ \sqrt{(1-\mu)^2 + R_f^2 + 2\mu/R_f + 2\,(1-\mu)/(1-R_f) - 2\,(1-\mu)\,R_f + J_t} \tag{A10}$$

$$\Delta V_{tot}^{min}(R_i, R_f, J_t) = -\left(\sqrt{(1-\mu)/R_i} - R_i + \sqrt{\mu/R_f}\right) + \sqrt{R_i^2 + \mu^2 + 2\,(1-\mu)/R_i + 2\,\mu - \mu\,R_i^2 + J_t}$$
$$+ \sqrt{(1-\mu)^2 + R_f^2 + 2\mu/R_f + 2\,(1-\mu) - R_f^2\,(1-\mu) + J_t} \tag{A11}$$

From Eqs. (A10) and (A11), one has

$$\Delta V_{tot}^{max}(R_i, R_f, J_t) - \Delta V_{tot}^{min}(R_i, R_f, J_t) \le f(R_i, R_f) \tag{A12}$$

where

$$f(R_i, R_f)$$
$$\triangleq \sqrt{\left|R_i^2 + \mu^2 + 2\,(1-\mu)/R_i + 2\,\mu/(1-R_i) - 2\,R_i\mu\right|} + \sqrt{\left|(1-\mu)^2 + R_f^2 + 2\mu/R_f + 2\,(1-\mu)/(1-R_f) - 2\,(1-\mu)\,R_f\right|}$$
$$- \sqrt{\left|R_i^2 + \mu^2 + 2\,(1-\mu)/R_i + 2\,\mu - \mu\,R_i^2\right|} - \sqrt{\left|(1-\mu)^2 + R_f^2 + 2\mu/R_f + 2\,(1-\mu) - R_f^2\,(1-\mu)\right|} \tag{A13}$$

Suppose, for instance, that $R_i \in [0.0170, 0.0426]$ DU (corresponding to $R_i \in [6545, 16{,}378]$ km) and $R_f \in [0.0048, 0.0305]$ DU (corresponding to $R_f \in [1840, 11{,}740]$ km). Then, $f(R_i, R_f) \le 7.33 \times 10^{-4}$ DU/TU (0.75 m/s). In other words, as long as $R_i, R_f \ll 1$, $f(R_i, R_f)$ is very small and the approximation $\Delta V_{tot} \cong \Delta V_{tot}^{max}$ is justified. Under such an assumption, Eq. (16) coincides with Eq. (A10).

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